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Minimal L_p Projections by Fourier, Taylor, and Laurent Series

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In appropriate function space settings, it is proved that the Fourier, Taylor, and Laurent series projections are minimal in all L_p norms ($1 \leqslant p \leqslant \infty$). This result unifies and extends known results for the Fourier, Taylor, and Laurent projections in L_∞ and for the Fourier projection in L_1. The proof is based on a generalisation of a kernel summation formula due to Berman.

1. INTRODUCTION

Consider the following general setting. Suppose that ν is a positive complete measure on a subset S of a domain D invariant with respect to some group T of automorphisms on S. For each t in T, let E_t be the isometry on the invariant (w.r.t. E_t) subspace X of $L_p(S, \nu)$, where $1 \leqslant p \leqslant \infty$, given by

$$(E_t f)(s) = f(t(s)) \qquad (f \in X, s \in S).$$

Suppose also that μ is some fixed complete measure on T with the property that

$$\int_T |d\mu| = 1, \tag{1}$$

and that a projection P_0 (i.e., a bounded, linear, idempotent mapping) is defined from X into a given subspace Y.

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In this general setting we shall prove that P_0 is a minimal L_p projection provided that a "generalised Berman kernel formula" holds, and hence that the classical Fourier, Taylor, and Laurent projections are minimal L_p projections, where, in the case of the Taylor and Laurent projections, L_p norms may be taken over either the complete domain (D) or an interior circular contour (Γ).

The Fourier, Taylor, and Laurent series projections fit into the general setting by making appropriate choices of D, S, T, X, Y, E_t, ν, μ, and P_0 as follows.

Case 1 (Fourier Projections). $D = [0, 2\pi]$, $S = D$, $s = x$, $T = D$,

X: 2π-periodic L_p-integrable functions of x normed in L_p on S,

Y: trigonometric polynomials of fixed order n in x (i.e., algebraic polynomials of degree n in e^{ix} and n in e^{-ix}), $(E_t f)(s) = f(s+t)$, $d\nu(x) = dx$, $d\mu(t) = (1/2\pi)\, dt$,

$P_0 f$: partial sum of order n of Fourier series of f.

Case 2a (Taylor Projection—Domain Norm).

D: $|z| \leqslant \rho$ $(0 < \rho < \infty)$, $S = D$, $s = z$, T: $|t| = 1$,

X: functions analytic in the interior of D and continuous on D normed in L_p on S,

Y: algebraic polynomials of fixed degree n in z,

$(E_t f)(s) = f(ts)$, $d\nu(z) = dS$ (element of area), $d\mu(t) = (1/2\pi i)(dt/t)$,

$P_0 f$: partial sum of degree n of Taylor series of f.

Case 2b (Taylor Projection—Contour Norm). The same setting is used as in Case 2a, except that

$$S = \Gamma\colon |z| = \rho_0 \ (\rho_0 \text{ fixed, } 0 < \rho_0 \leqslant \rho),\ d\nu(z) = |dz|.$$

Case 3a (Laurent Projection—Domain Norm).

D: $\rho_1 \leqslant |z| < \rho_2$ $(0 < \rho_1 < \rho_2 < \infty)$, $S = D$, $s = z$, T: $|t| = 1$,

X: functions analytic in the interior of D and continuous on D normed in L_p on S,

Y: algebraic polynomials of fixed degrees n in z and m in z^{-1}, $(E_t f)(s) = f(ts)$, $d\nu(z) = dS$ (element of area), $d\mu(t) = (1/2\pi i)(dt/t)$,

$P_0 f$: partial sum of degrees n in z and m in z^{-1} of Laurent series of f.

Case 3b (Laurent Projection—Contour Norm). The same setting is used as in Case 3a, except that

$$S = \Gamma: |z| = \rho_0 \ (\rho_0 \text{ fixed}, \ \rho_1 \leqslant \rho_0 \leqslant \rho_2), \ dv(z) = |dz|.$$

2. GENERALISED BERMAN KERNEL FORMULA

LEMMA 2.1. *For the Fourier, Taylor, and Laurent projections P_0, in the setting of Section* 1,

$$\int_T (E_t^{-1} P E_t f)(s) \, d\mu(t) = (P_0 f)(s), \tag{2}$$

for an arbitrary projection P of X into Y.

This formula was obtained first for the Fourier projection by Berman [1] and, subsequently, for the Taylor projection by Geddes and Mason [2], and for the Laurent projection by Mason [3]. In each case the formula may be deduced by showing it to be exact for all elements in a basis for X, namely,

$$\begin{array}{lll} \{e^{ikx}\} & (k = 0, \pm 1, \pm 2, \ldots) & \text{in Case 1,} \\ \{z^k\} & (k = 0, 1, 2, \ldots) & \text{in Case 2,} \\ \{z^k\} & (k = 0, \pm 1, \pm 2, \ldots) & \text{in Case 3.} \end{array}$$

3. MINIMAL PROJECTIONS

THEOREM 3.1. *Consider the setting of Section* 1 *for any fixed p* $(1 \leqslant p \leqslant \infty)$ *and suppose that the generalized Berman kernel formula* (2) *holds for a given projection P_0 and an arbitrary projection P of X into Y. Then P_0 is a minimal projection in the L_p norm on S.*

COROLLARY 3.2. *In the setting of Section* 1, *the Fourier, Taylor, and Laurent projections are all minimal L_p projections. In the case of Taylor and Laurent projections, the L_p norm may be measured over either the domain D or the interior circular contour Γ.*

Proof of Corollary 3.2. This follows immediately from Lemma 2.1 and Theorem 3.1.

Proof of Theorem 3.1. Suppose that $(1/p) + (1/q) = 1$, so that $\|\cdot\|_p$ and $\|\cdot\|_q$ are dual. Then

$$\|f\|_p = \sup_{\|h\|_q = 1} \int f \cdot h \, dv. \tag{3}$$

Now

$$\|P_0 f\|_p = \left[\int_S \left|\int_T (E_t^{-1}PE_t f)(s)\,d\mu(t)\right|^p dv(s)\right]^{1/p} \qquad \text{from (2)},$$

$$\leqslant \left[\int_S \left\{\int_T |(E_t^{-1}PE_t f)(s)| \cdot |d\mu(t)|\right\}^p dv(s)\right]^{1/p}$$

$$= \left\|\int_T |(E_t^{-1}PE_t f)(s)| \cdot |d\mu(t)|\right\|_p$$

$$= \sup_{\|h\|_q=1} \int_S \int_T |(E_t^{-1}PE_t f)(s)| \cdot |d\mu(t)| \cdot h(s)\,dv(s) \qquad \text{by (3)}.$$

Reversing the integration (by Fubini's theorem):

$$\|P_0 f\|_p \leqslant \sup_{\|h\|_q=1} \int_T \int_S |(E_t^{-1}PE_t f)(s)| \cdot h(s) \cdot dv(s) \cdot |d\mu(t)|$$

$$= \int_T \sup_{\|h\|_q=1} \int_S |(E_t^{-1}PE_t f)(s)|\,h(s)\,dv(s)\,|d\mu(t)|$$

$$= \int_T \|(E_t^{-1}PE_t f)(s)\|_p\,|d\mu(t)| \qquad \text{by (3)},$$

$$\leqslant \int_T \|E_t^{-1}\|_{L_p} \|P\|_{L_p} \|E_t\|_{L_p} \|f\|_p\,|d\mu(t)|$$

$$= \int_T \|P\|_{L_p} \|f\|_p\,|d\mu(t)| \qquad \text{since } \|E_t^{-1}\|_{L_p} = \|E_t\|_{L_p} = 1,$$

$$= \|P\|_{L_p} \|f\|_p \qquad \text{by (1)}.$$

Hence $\|P_0\|_{L_p} = \sup_{\|f\|_p=1} \|P_0 f\|_p \leqslant \|P\|_{L_p}$, and the theorem is proved. Q.E.D.

These results were previously known only for particular norms. Specifically, results in L_∞ were given for Fourier, Taylor, and Laurent projections, respectively, by Lozinski [4], Geddes and Mason [2], and Mason [3]. (There is no proof in Lozinski's paper, but one is given by Cheney [5].) The corresponding Fourier result in L_1 was pointed out by Lozinski [4] and a proof in a more general setting was given by Lambert [6].

4. Related Results

Uniqueness questions are not considered here, but some results are already known. The Fourier projection on trigonometric polynomials is the unique minimal projection in L_∞ (Cheney *et al.* [7]) and in L_1 (Lambert [6]). Moreover, these results hold true for spaces of complex-valued functions as well as for spaces of real-valued functions.

Generalisations are also possible. For example, Lambert [6] has generalised the Fourier results in the context of compact Abelian groups. Moreover, all the results of Sections 2 and 3 may be generalised readily to multivariate functions on hypercubes (for real variables) or on polydiscs and polyrings (for complex variables), and a detailed discussion of this topic in L_∞ and L_1 norms is given by Mason [8].

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